Human stochastics: Unconscious, ratiomorphic processes as the foundation of perception and learning

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ABSTRACT

The purpose of this paper is to demonstrate how unconscious-ratiomorphic processes are fundamental for all perceptions and learnings in general. This is especially relevant due to artificial intelligence (AI) is presenting us with new challenges. So, the "end of all theory" has even been announced as a provocation to scientific theory — because ample correlations are supposedly sufficient from a pragmatic point of view. Is this really a loss? The answer is: No, because there have never been real certainties in the philosophical sense. However, people generally find uncertainty unpleasant, as psychology and cognitive science prove. There are theoretical and practical reasons: If we dispense with "causality", methodological abysses open up. Our everyday life also relies on causality, because responsibility seems impossible without it. So, is "technical stochastics" incomprehensible, useless or even dangerous in the age of AI? No, because we can't model cognitive reality using only conscious-rational processes (as demanded by epistemological rationalists). However, perception and learning are based on unconscious-ratiomorphic processes — what we call "human stochastics". Humans are not passive beings. Rather, they constantly carry out active inferences based on epistemic actions. The brain uses Bayesian statistics to minimize "prediction errors". As a "predictive mind", it uses a hierarchical multi-level model to simultaneously examine invariances at various granularities — and thus improves the prediction of action effects. Perception and learning are understood as stochastic processes. Taken literally, "human stochastics" is unavoidable, childishly simple and commonplace. Hence, with "human stochastics" we can model the "actual genesis" of percepts and learning dynamics in detail.

Keywords: *Cognitive modelling, Abduction, Epistemic actions, Active inference, Bayes, Pragmatism.*

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Highlights of this paper

- Firstly, we demonstrate that purely deductive-logical research is a myth, since abductive and qualitative-inductive processes are always necessary for new knowledge to emerge.
- Further, we argue how unconscious-ratiomorphic processes underpin both everyday perception and logical reasoning in mathematics.
- Finally, it is made clear that cognitive modelling requires an active observer with epistemic actions, because the cognition of a "predictive mind" aims to minimize "predictive errors".

1. INTRODUCTION

1.1. A Paradigm Shift Driven by Artificial Intelligence?

The current triumphant advance of artificial intelligence (AI) is causing remarkable unrest in everyday discourse, as illustrated by the examples listed by Russel [1]. However, the scientific community is also alarmed — see Gethmann, et al. [2]. This is because the methods of machine learning and big data not only influence the statistical possibilities for researching correlations with great efficiency: The scientific-theoretical question of what a "theory" is in the first place and whether correlations are "sufficient" or only necessary for this must be asked anew. Stochastic aspects are of some interest in the philosophy of science. But what is even more striking is that despite the high topicality of AI (and the stochastic methods it uses), stochastics itself does not seem to be gaining in popularity. What could be the reason for this? Do people in general have an objective relationship with uncertainty and indeterminacy — or is it rather characterized by emotions?

2. PEOPLE'S DISCOMFORT WITH UNCERTAINTY

2.1. Uncertainty from the Perspective of Cognitive Psychology

AI puts its finger in the wound that natural intelligences sometimes don't like to admit: Most people feel uncomfortable with uncertainty and indeterminacy. On the one hand, there are evolutionary reasons for this, as Lotto [3] emphasizes: In prehistoric contexts, uncertainty about the supply of water could be just as deadly as uncertainty about whether the shadow in the bushes might be a predator. The aversion to uncertainty and indeterminacy can also be demonstrated experimentally. Lotto [3] for example, reports on experiments with artificial living systems. The artificial organisms developed differently depending on their environment: in an "uncertain environment (with hardly predictable stimuli and therefore frustratingly unsuccessful predictions), they tended to show environment-dependent (i.e. experience-based) behaviors. If, on the other hand, the retina was able to develop in an unambiguous environment (with a balanced relationship between stimulus and reward), the neuronal processes were also significantly more complex. There, the artificial retina developed different types of receptors, which can be interpreted as a precursor for color vision. Lotto [3] concludes from the results of this study:

"Overcoming uncertainty and predicting usefully from seemingly useless data is arguably *the* fundamental task that human brains, as well as all other brains, evolved to solve … hence why existing in uncertainty is exactly what our brains evolved to *avoid*. Living systems in general hate uncertainty."

For the cognitive neuroscientist Karl Friston, reducing uncertainty is actually the most important function of the brain — see Peters, et al. [4] and Parr, et al. [5]. Accordingly, the brain's measurable ability to make accurate predictions is of central importance. This gives rise to the title of Hohwy [6] which can be read as an introductory textbook to Karl Friston's complex theory: *The Predictive Mind.*

Dörner [7] speaks of a certainty drive, which is to be understood as the motivational basis for a research drive and research is nothing other than the attempt to reduce uncertainty (p.353). In theoretical psychology,[1] Dörner [7] models the need for determination of his simulated organisms as a homeostatic concept: If a defined level of *certainty* is undercut, the behavioral program *exploration* starts with the aim of reducing the uncertainty. Indeterminacy thus corresponds to a "state of deficiency" (p.359), which can be remedied by exploration, cf. Bach [8]. A central variable for the success of the behavioral program *exploration* is the so-called *expectation horizon*. Here, the AI (the simulated cognitive system) extrapolates those events into the future that it knows (or assumes) are in progress (p.362). At this point at the latest, it becomes clear how similar the approaches of Dörner [7] and Parr, et al. [5] are. This is because *The Predictive Mind* is precisely what Dörner models here as the intelligent performance of his artificial being (even if Dörner is also strongly interested in emotional and motivational aspects, which is not the case for Karl Friston and colleagues).

2.2. Philosophical Perspectives on Uncertainty

It is now astonishing that philosophy also has an at least divided relationship to the topic of probabilities, which has become clear time and again over the last 275 years and continues to have an effect to this day. Gerhard [9] for example, says that the impossibility of justifying the principle of induction is "unsatisfactory". He is referring here to the still unsolved induction problem of David [10]. Hume was the first to explicitly point out that any learning from experience always presupposes that learning by means of inductive inferences leads to valid conclusions. For example, the general knowledge that the sun will rise again tomorrow morning always presupposes that nature is uniform. But how can this be known with certainty? On closer inspection, the hypothesis that the sun rises every day is itself the result of an inductive learning process. Learning by means of inductive generalization therefore always presupposes the success of the inductive learning processes — without being able to justify this in any other way. Consequently, every inductive *truth* is actually based on an *infinite regress* or circular reasoning. This is precisely what is unsatisfactory, says Gerhard [9]. This is because the truth in particular, when a general pattern is inferred from a number of individual cases, is in reality only a probability. This means that there are no real certainties in the world, only probabilities.

Few philosophers tackle the problem of induction as aggressively as Mill [11] who also wanted to rethink mathematics on an empirical basis. For him, it is sufficient if mathematics proves itself sufficiently empirically. van Fraassen [12] takes a similar position, rejecting the concept of truth in his *Constructive Empiricism*. Instead, he proposes replacing truth with *empirical adequacy* (empirical appropriateness). This should keep the fundamental falsifiability in mind: Every statement and every theory is therefore not true for all time. Rather, it is confirmed with sufficient probability with regard to a specific test situation. For many other philosophers, such as Ludwig [13] this all too uncertain foundation is not sufficient to fulfill their claims to certainty. Stephen [14] provides an overview of the problematic history of objectivity. And Thomas [15] points out that there are good reasons to want objective certainty:

"The previous considerations show that an objective understanding of truth is a prerequisite for the ability to criticize one's own worldview or the worldviews of others. Without objective truth, such world views and the practices associated with them are immune to criticism. It is then difficult to object to female circumcision in East Africa or the rational Nazi. It has also been shown that living in truth is not only useful, but also intrinsically valuable for us."

[1] Bach [8] can serve as an English introduction to the theory of Dörner [7] who develops a comprehensive approach to "synthetic intelligence".

2.3. The Induction Problem as a Phenomenon of Uncertainty

This instrumental and intrinsic need for objective truth has led us to simply *forget* the unresolved induction problem in everyday life and act as if it had been solved — or as if it had never arisen. This forgetting or ignoring would thus be an auxiliary construction in the sense of Hans [16]. Vaihinger explains how a multitude of counterfactual constructions ensure that practice in everyday life and science functions at all. This applies equally to cognitive and social practices, which, due to their circularity or infinite regress, would otherwise run as infinitely continuous processes. Accordingly, Siegfried [17] and Siegfried [18] refers to these constructs as *process interrupters.* For even an empirical measurement would by no means bring about an indisputable interruption of the process, as Peter [19] argues: Since "every measurement result already has the logical structure of an if-then proposition, in which the conditions of the undisturbed functioning of the devices used are determined in the if-part, and the measurement result in the then-part. [...] Empiricism alone does not distinguish the disturbed from the undisturbed device [...]."

From a strictly logical point of view, we cannot escape the induction problem and all the resulting uncertainties. The as-if construction of an objective truth serves recognizably purely pragmatic reasons, as Peter [19] suggests in a pragmatist manner. This is because the truth as a *means* serves identifiable *purposes* — and in the sense of Lotto [3] quoted above, this is simply the survival of mankind. Accordingly, the discomfort towards the indeterminate and the uncertain is existentially underpinned. But must this unease also be transferred to the theoretical examination of uncertainty? Against this background, is there a humane mathematics, especially a humane stochastics? Does it have to exist because of the relevance of stochastics for humans in the century of artificial intelligence? Or is the enemy image of stochastics in the school hate subject of mathematics completely unfounded? Would stochastics be more than exciting if only the relationship to humans were clearer? This perspective will be outlined here. But first: What do we want to differentiate ourselves from?

3. TECHNICAL STOCHASTICS IN THE AGE OF AI: INCOMPREHENSIBLE, USELESS OR EVEN DANGEROUS?

3.1. The Discomfort with Mathematics and Stochastics

For pupils in Germany, there are apparently two classic stereotypes of aversion: Firstly, the hate subject of mathematics in general and then the enemy image of stochastics in particular. This is suggested by Martin [20]; Iris [21] and Marco [22]. However, the phenomenon is apparently not limited to Germany, asEdeh [23] for Nigeria and Natasha [24] for Pakistan show by way of example. Gary [25] reports that 75% of Americans opt out of math and stay away from many math-related professions. He cites fear of math as one of the main reasons for this. AndCaitlin [26] claims in *The Harvard Gazette*: "Math anxiety is worldwide and very, very real." Szucs and Toffalini [27] show that this anxiety is not based on objective reasons, but that subjective factors such as expectation of success, self-efficacy and perception of control play a central role.

3.2. Current Scientific-Theoretical and Methodological Problems

The fear of mathematics in general is therefore very widespread. But where does the unease about stochastics in particular come from? For reasons of space, we will limit ourselves here to the connection between artificial intelligence and stochastics. Artificial intelligence is currently causing considerable unease in the philosophy of science in particular, as Gethmann [28] documents: "While the sentence *All swans are white* would be easy to falsify, the sentence *Probably all swans are white* raises considerable problems."

Accordingly, a stochastic understanding of "truth" shows us the limits and problems of falsification as such. This is no small matter, because since Karl [29] our understanding of science has used *falsifiability* as a central criterion for distinguishing it from everyday communication and pseudoscience. However, this strict concept of science is only possible where causal explanations in the narrower sense are concerned. Essentially, these are the natural sciences such as physics and chemistry. In the humanities, social and cultural sciences, on the other hand, such a causal explanation is usually not possible due to the complexity of the influencing factors. There, arguments are very often based on statistical correlations because the causal interpretation of an individual case is generally not available.

Nevertheless, causal modelling remains the ideal of all sciences, which is why Norbert [30] advocates a distinction between *strong causality* (as is typical of the natural sciences) and *weak causality* (as is characteristic of complex systems) — see also Jan [31]. It is revealing that Norbert [30] interprets the humanities, social sciences and cultural studies as *structural sciences*. Bischof sees biology as a prototypical structural science in clear methodological contrast to physics, which he regards as the archetype of the *materials sciences*. Like physics, material sciences are suitable for a mathematical approach that can represent causality quantitatively. As Norman [32] points out, this goes so far that physics and mathematizability can be equated in terms of definition. In contrast, Norbert [33] recommends a methodology for structural sciences that is more qualitatively oriented.

A cybernetic system theory would do this optimally, because in the first step it outlines a qualitative model as a structure of effects. In further methodological steps, this model can be differentiated as desired (e.g. by analyzing a *black box* in the block diagram with finer granularity and thus becoming a *white box* because the logical structure inside it becomes visible). Depending on the interest in knowledge, a quantitative analysis is possible, but not absolutely necessary. Accordingly, there are different ways of falsifying such a theory, as Norbert [33] differentiates: "In contrast to the prevailing understanding of the experiment in psychology as a test of a hypothesis preferred by the investigator, cybernetic system analysis is usually about deciding between several equally valid alternatives." This logical-qualitative approach thus corresponds to the motto of the mathematician Haftendorn [34]: "It is better to understand without calculating than to calculate without understanding."

3.3. The »End of Theory« as A Scientific Utopia

The spectacular "end of theory" outlined by Anderson [35] in his widely acclaimed article propagates a completely contrary position. Gethmann [28] summarizes his approach: By means of *big data* and its processing by AI, the sciences would be able to dispense with the difficult business of causal explanation — and replace it with correlation analyses. In this sense, one could speak of an end of theory. However, the demise of theory would not mean the loss of something valuable. Rather, an annoying, actually impracticable task, namely the explication of an adequate understanding of causality, could finally be abandoned.

This is where an intellectual reluctance arises, because our understanding of science goes far beyond establishing correlations. As I said, *causal modelling* is the ideal of all science. However, causal modelling is not a value in itself, but a methodological attitude. In essence, it is about the distinction between modern and ancient-medieval conceptions of science. Gethmann [28] sums up this difference as "contemplation versus intervention". Accordingly, a contemplative style observes what happens to be given and ponders its meaning, as we have known it since Plato. Francis Bacon conceives of ideal science as an empirical method in a completely different way. As Gethmann [28] explains, his interventionist style of cognition aims to master nature by uncovering cause-effect relationships. This involves intervening in nature in a planned manner, which is intended to maximize the cognitive value as well as the application value of a theory.

In this sense, AI-supported research falls behind modern empiricism because it only relies on correlations. In the worst case, these are only coincidences that do not contain any causal aspects. However, this is not the only source of unease towards a science that has resigned itself to an end of theory. In addition to intellectual and scientific-theoretical criticism, we must also remember people's discomfort with uncertainty (see section 1).

This discomfort is accompanied by a second fear when we think of the application value of any science. The application of AI in particular harbors certain risks that are not obvious (in contrast to the dystopias of AI taking over the world). This is a problem that is inherent in all statistics. The basic problem can be illustrated with a very simple example: Let's imagine an ordinary family with two children, aged seven and ten. Then the four people would typically have shoe sizes 31, 35, 39 and 43. The average of these numbers would be 37 and it immediately becomes clear that shoes with this average shoe size would not really fit anyone in this household. There is simply no one here who is a size 37.

3.4. Methodological Artifacts as A Normative Threat

However, it becomes problematic when *descriptive* values are ascribed the normative status of *prescriptive* target values. This is the case, for example, when deviation from such a pseudo-norm is discriminated against and sanctioned. Unfortunately, this happens comparatively often, cf. Jürgen [36] and Lischka and Klingel [37]. In this case, a descriptive indicative becomes a prescriptive imperative.

In the above example with shoe sizes, this leads to obvious nonsense when size 37 is elevated to a desired target value. It is not desirable for the individual to submit to this standard (although there was an effective standard for the size of women's feet in China for centuries, which was also enforced by deforming bandages). It therefore makes little sense to only produce size 37 shoes on an industrial scale (although this would reduce production and storage costs, among other things). A similarly prescriptive imperative was hidden for a long time in the concept of the body mass index (BMI), to which people were supposed to subordinate themselves — see McKenzie [38].

Such prescriptive imperatives can be overt or covert. Imperatives such as "Be like everyone else!" are often placed unconsciously in education, so that this remains just as hidden from the educators as from the addressees of the messages. This is because it is very easy to derive a *should be like* from a random *is like*. And this is very often done in everyday life (e.g. when a child is told to behave like the others at school, or when a child is expected to choose the parents profession, etc.).

However, an *ought* can never be logically derived from a *being*. David [39] (there in section 3.1.1) had already proven this. Consequently, methodically produced values cannot claim to be valid as setpoint values. At first glance, such methodological artifacts sometimes appear to be descriptive sentences, but they are not. In reality, they are prescriptive propositions that spring from a power-political agenda. Several distortions are made here — consciously or unconsciously. It is not only the pretense of a normative power of the factual that is questionable here. Another arbitrary act is the selection of the data that is processed (*data bias* or *sampling bias*). The choice of method also has a major influence on the results that will emerge (*algorithmic bias*). These and other biases are systematically presented in the context of machine learning by Humm, et al. [40]. However, they apply to learning in general.

3.5. Manipulation by Actively using Biases

Through a virtuoso application of such distortions, statistics can be produced as desired in order to communicate the preferred opinion in the next step. Many examples of this can be found in Bosbach and Korff [41]; Dubben and Beck-Bornholdt [42] and Walter [43]. Here, methodological artifacts are produced whose normative validity is simply asserted for a group of people. The self-portrayal of people in social media can also serve as a current example, where (partly consciously and partly unconsciously) methodical distortions are used to present oneself as bigger than life. These ultimately fictitious facts have an effect on natural persons, who often accept such exaggerated presentations as the new norm — see Vogel, et al. [44].

If such a claim is actually accepted, this claim to validity does not necessarily have to be consciously reflected upon. The unnoticed control of actions through *nudging* is a much-discussed application scenario for big data and deep learning, see Passoth and Straßheim [45]. However, the benevolent influencing of behavior also constitutes manipulation. As Gethmann [28] critically notes, the loss of the individual as the responsible originator of action is certainly an option.

The loss of the individual is also to be understood as a loss for the individual. This is because the normative act deprives the specific person of specific opportunities. This can be a lack of options, for example because a loan is not approved due to normative attributions. Such *normalism* also delegitimizes one's own feelings, because it is no longer one's own perception that counts, but the set norm. This is true even if the norm is a methodological artifact that has no rational basis. For example, this can be the reduction of a person to a single characteristic (such as gender, ethnicity, profession, etc.). From a cognitive-semiotic perspective, this is massively undercomplex, because every semantic aspect is based on a pragmatic context of action. Accordingly, the semantic potential is as diverse as the conceivable uses of an object — see Schwarzfischer [46]. This also applies to people as subjects insofar as they take on completely different social roles in different contexts — and thus have a differential semantics that depends on the respective pragmatics. And it is precisely this potential diversity that can be curtailed by a normative attribution. It is therefore by no means irrational for people to fear inhuman statistics.

As a *social system*, every authority must handle a clear guiding distinction, as Niklas [47] demonstrates. Accordingly, the school system does not simply reflect the bio-psycho-physical reality. Rather, the school system looks through its *administrative-legal glasses* and abstracts from everything that does not correspond to the guiding distinction. This is exclusively about *exam passed* versus *exam failed* — and not about whether a child is intelligent or not. It is not about persons, but exclusively about roles in the social system. To ensure this, reality is reduced to a partial reality that only follows the binary code of the respective sub-system. A school system must therefore operationalize a clear guiding distinction. The unambiguousness of right and wrong can be described as the basis of our school system. As part of an administrative system, legal certainty is also taken into account. It is therefore documented that every operation follows the legal guiding distinction of *lawful* versus *unlawful*. Just as the scientific system is only interested in whether something is *true* or not. Everything else is ignored and is not part of the scientific system.

According to this logic, stochastics is not presented at school in such a way that it is as interesting and stimulating as possible for the pupils. On the contrary, the school as a functional system optimizes its own operations by shortening stochastics (like all other subjects) to a binary distinction. This makes teaching easier for teachers, makes it easier to correct examinations and also secures the system in terms of administrative law.

4. HUMAN STOCHASTICS: TAKEN LITERALLY, IT IS UNAVOIDABLE, CHILD'S PLAY AND COMMONPLACE!

4.1. Stochastics as the "Art of Conjecture" or "Guesswork"

It would be possible to experience stochastics completely differently from the way it was presented in section 3 — see Gigerenzer, et al. [48]. This is because the etymological root of stochastics is found in Greek as *stochastikē technē*. And that means something like art of *conjecture* or *guesswork*. This section will outline how topical this ancient interpretation of the term could be. To do this, however, we must first distance ourselves from the usual view of stochastics. In the school context, usually a special case is discussed that does not correspond to the general principle. The examples of conventional games of chance in particular do not even peripherally touch on the relevance of everyday stochastics. It is therefore no wonder that pupils tend to feel bored by this. After all, their lives are generally not influenced by this form of gambling. However, it would be possible to approach stochastics using examples of everyday life that connects the people of a culture. The approach of an *art of guessing* could serve as a foundation here.

As will be shown below, this *art of conjecture* would also be a hinge to other philosophical perspectives (such as epistemology and philosophy of science). This is all the more true the less these disciplines operate in a vacuum. In plain language, this means that an *empirical subject of knowledge* must represent the center (i.e. a living human being) instead of always presupposing an *ideal subject of knowledge* as a methodological artifact (as a purely rational subject that indulges in pure contemplation without emotion or history and without socio-cultural embedding). The real human being is an acting subject and not a passive-contemplative being, as Schwarzfischer [46] demonstrates. Accordingly, cognition is not only produced in a passive-contemplative way; rather, *epistemic actions* must be understood as the prototype of an active-ideomotor cognition. This means that cognitive processes are always embodied and situated. This means that cognitions can be analyzed with different levels of resolution (granularity) in terms of time, space and content.

4.2. The "Inquiry Cycle" Consists of Abduction, Deduction and Induction

Dewey [49] illustrates this excellently — especially in the visualization according to Jörg [50] which can also be found in Schwarzfischer [46]. There it becomes clear that every problem solution consists of six sub-processes that have very different functions:

1. Some irritation is noticed without it being clear what it is.
2. A tentative definition of the problem is created.
3. A hypothesis is formulated as to what could have led to the disturbance.
4. Options for testing this hypothesis are derived.
5. The hypothesis is tested for viability by experimental action (or mental rehearsal).
6. The ability to act is restored — otherwise the *Inquiry Cycle* is repeated.

As can easily be seen, the functions of these sub-processes can be linked to aspects of scientific theory. The *Inquiry Cycle* combines those aspects that Hans [51] referred to as the *context of discovery* and the *context of justification*. Like Karl [29] also Reichenbach was of the opinion that the circumstances of the discovery of a connection are only of psychological interest and are therefore irrelevant in terms of philosophical theory of science. This applies as little to epistemology as it does to the didactics of mathematics. This is because discovery is an elementary sub-process of the inquiry cycle and thus also of any *discovering learning* in mathematics didactics, cf. Michael [52].

If we understand the sub-processes of the Inquiry Cycle as logical operations, it quickly becomes clear that a third type must be added to the familiar deduction and induction. *Deduction* does not create anything new because it proceeds analytically in the strict sense. This means that it only reveals those facts that are already contained in things. *Induction* does not produce entities either, but provides an assessment of the plausibility of assumptions. It methodically leads to confirmations or refutations of hypotheses — but induction itself does not produce any new hypotheses! Only *abduction*, which was systematically introduced by Charles [53] produces new hypotheses. The *Inquiry Cycle* according to Dewey [49] combines the processes of abduction, deduction and induction into a pragmatic whole for the investigation of problems. *Abduction* plays a central role in this. Therefore, it can really be understood as the *art of conjecture*, without which mathematical thinking would be impossible, as Michael [52] argues.

4.3. Abduction as an Alleged Syllogism

Methodologically, *abduction* can be described as a third form of logical reasoning. But let's start with deduction and induction in order to better classify the third form — see Susanne [54]; Jo [55]; Michael [52] and Dominik [56].

Deduction is the oldest form of reasoning that has been explicitly discussed in epistemology since antiquity. Aristotle used the term *syllogismos* to describe a deductive argument, which he was the first to define: "A deduction (syllogismos) is therefore an argument in which, if something has been posited, something other than the law necessarily follows from the law".[2]

In the following, we take the example of Charles Sanders Peirce.[3] He uses this to run through all three combinatorial variants. This should show that, in addition to deduction and induction, there must be a third logical form, which he calls abduction. For *deduction*, the example has the form:

Rule (major premise):	All beans from this bag are white.
Case (minor premise):	These beans are from this bag.
Result (conclusion):	These beans are white.

This conclusion necessarily follows from the fact that the upper and lower propositions (the major premise and the minor premise) are assumed to be certain knowledge. David [39] already doubted that such certain knowledge could exist at all (because it would itself have to have arisen from inductive learning).

Induction does not presuppose these certainties, but is intended to produce knowledge that is as certain as possible in the first place.[4] However, the results of this form of inference are not necessarily true, instead they are only more or less probable. For induction, our example takes this form:

Case (major premise):	These beans are from this bag.
Result (minor premise):	These beans are white.
Rule (conclusion):	All beans from this bag are white.

The form of the syllogism already suggests that a third variation is possible, in which the *case* is now to be identified as the conclusion. Accordingly, the *rule* and the *result* now serve as upper and lower premises. For the *abduction*, our example therefore has the form:

Rule (major premise):	All beans from this bag are white.
Result (minor premise):	These beans are white.
Case (conclusion):	These beans are from this bag.

Abduction is often introduced in this form after Charles [53] himself demonstrated it in this way (in his *Collected Papers* CP 2.623). However, he allowed himself to be seduced to a certain extent by the combinatorial form

[2] Aristoteles: Topics I 1, 100a25-27

[3] The bean example of abduction from the Peirce Collected Papers (CP 2.623) originally comes from Charles [53]. However, Peirce still used the term *hypothesis* to describe abduction at that time. Cf. Charles Sanders Peirce (1931–1958): *Collected Papers.* [CP: Volumes 1–6, Hartshorne, Charles & Weiss, Paul (Eds.); Volumes 7–8, Burks, Arthur W. (Ed.)] Cambridge: Harvard Univ. Press. (The citations refer to the volume and the paragraph number, which are separated by a period.)

[4] *Inductive reasoning* should not be confused with the *principle of complete induction* in classical mathematics.

of the three propositions. As Jo [55] explains in detail, Peirce dealt with the problem of abduction for many years. (In his early writings, he still uses the word *hypothesis* and only later refers to it as *abduction*).

4.4. Criticism of the Syllogistic form of Abduction

What is exciting is that the form of syllogism quoted here is by no means the general principle of an abduction. Rather, it merely takes the existing tripartite scheme and creates the missing third variation. In the sense of *diagrammatic thinking* according to Peirce [57] the existing scheme is used to carry out (conscious or unconscious) thought experiments on it:

"By diagrammatic reasoning, I mean reasoning which constructs a diagram according to a precept expressed in general terms, performs experiments upon this diagram, notes their results, assures itself that similar experiments performed upon any diagram constructed according to the same precept would have the same results, and expresses this in general terms."[5]

Thus, the existing structure (the three-sentence scheme consisting of upper clause, lower clause and conclusion) suggests a mental restructuring of this very scheme. This is combinatorially correct, but epistemologically inappropriate. Strictly speaking, the new syllogism does not explain anything. Nor does it clarify the general principle of abduction. But what would be the general principle of an abduction? And why is it so important for understanding the art of conjecture?

It is advisable to distinguish between *qualitative* induction and *quantitative* induction — precisely because qualitative induction is often confused with abduction, as Jo [55] makes clear: "Qualitative induction thus infers the case from two known quantities, namely the result and the already known rule, and this is decisive." The situation is different with abduction, where no rule is yet known (and certainly not as an upper premise). As Reichertz [59] emphasizes, abduction is the only form of inference in which new schemata or hypotheses are generated: "The third (confusingly similar to qualitative induction, but nevertheless completely different) type of data processing now consists of compiling or discovering such combinations of characteristics based on the interpretation of the collected data for which no corresponding explanation or rule can be found in the existing knowledge stock."

Abduction is therefore the only cognitive process that generates a hypothesis that was not previously available to the subject. This is illustrated by Umberto [60] as a thought experiment in his book Kant and the Platypus, where the unknown, egg-laying, furry animal with a duck's beak stubbornly refuses to be categorized in known categories — and therefore requires abduction. An example from medicine can illustrate this: When a doctor diagnoses the most likely disease based on symptoms, he selects from the repertoire of those diseases that are known to him. Precisely this is qualitative induction. An abduction only takes place when a completely new disease is diagnosed that did not exist before. The (implicit or explicit) assertion of novelty is to a certain extent a defining component of abduction. For abduction produces a hypothesis — and only a hypothesis. This means that it is completely unclear at this stage whether this hypothesis is viable or whether it will later turn out to be a so-called dead duck. The collection of evidence to confirm or refute this abduction hypothesis represents the process of (quantitative) induction. These are two isolated processes that are easy to distinguish once the principle has been understood.

[5] Peirce [57] Further explanations on this can be found in Bauer and Ernst [58] although the quote is incorrectly reproduced there: Instead of "precep"" (for predetermination, arrangement), it incorrectly says "percept" (for perception, sensation).

4.5. Abduction and Induction in the Learning Process

The concept of learning can be traced back historically to Plato or Aristotle, as Winfried [61] documents. Platonic learning corresponds to a recollection of previously (in a previous life) already known facts. For the specific subject of cognition (as an individual human being), this fact is new, although this fact is already in the world. This corresponds roughly to the relationship between learner and teacher in modern conceptions — where something is new for the learner, although it is of course already known to the teacher. The fact to be learned is therefore already in the world. Aristotelian learning, on the other hand, is conceived as the discovery of something new in the world. This is because the subject of knowledge is confronted with a new fact without the assertion of a migration of the soul. Accordingly, learning is not conceived as recollection, but as the imprinting of sensory impressions on a soul that is conceived as a blank slate.

In both cases, a new cognitive schema appears that was not previously available to the individual cognitive subject. The new appearance of a cognitive schema is the extreme case of the mere change of a schema, which Piaget [62] famously referred to as *accommodation*, see Piaget [62] and Piaget [63]. Because the adaptation of schemata always has a trial and provisional character, the connection between accommodation and abduction is a close one, as Von Glasersfeld [64] puts it: "I see abduction as an integral part of accommodation." This means that every accommodation (i.e. every process of understanding that changes the cognitive model or cognitive schema) contains a speculative moment. At least one aspect is therefore based on assumptions that have not yet been proven. And with a certain probability, such abductive hypotheses will actually prove to be wrong after they have been tested with inductive processes.

Even in the simple Gestalt perception of form, every understanding already goes beyond the sensually given, as the following example illustrates:

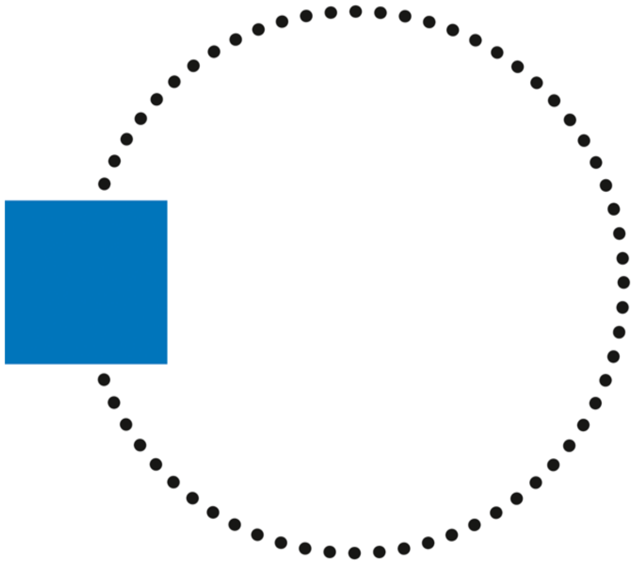

Figure 1. A "square" covers part of a "circle".
Source: Schwarzfischer [46].

In Figure 1 we immediately see a small square covering part of the larger circle — at least we think we see it *immediately*. Although we think we see it immediately, it is the result of quite complex unconscious processes. What microcognitive processes are involved specifically when we look at Figure 1 ? The circle is first constructed by the observer in the process of perception because, strictly speaking, only small dots fall on the retina in a certain arrangement.

Due to *invariances*, this is interpreted as a dotted line, which in turn is perceived as a contour. However, these invariances are not direct perceptions, but interpretations formed on the basis of previous experience. Basically, it is a hypothesis, but not an abduction: The hypothesis turns out to be a qualitative induction. This is because we have been confronted with the same problem many times before and are therefore already familiar with this type of solution.

From birth, we handle all accessible objects and realize that many of them can be moved — and remain consistent. The object therefore remains stable in itself, i.e. the object is invariant to displacement. It is precisely this translational symmetry that is the basis for us speaking of an *object* at all. For Figure 1, this means that the individual points of the dotted circle line do not behave like unconnected beads. Instead, the circle is expected to react as a whole when we grasp and move a part of it. The same applies to the square, which we expect to represent an equally consistent and stable object. We tried this ourselves hundreds of times in early childhood with all tangible objects. This is the only reason why we can assume that it is such a general principle that it will also apply to the objects in Figure 1.

The asserted validity is a hypothetical generalization based only on probabilities. There can be no talk of logical necessity here. This is characteristic of the processes of perception and categorization (which are usually defined as perceptual judgments). They use similarities to form categories. But how similar is similar enough? *Similarity* is obviously a gradual concept that must be understood as a continuous measure of invariance — see Zabrodsky, et al. [65] and Zabrodsky, et al. [66] and Zabrodsky and Algom [67]. Similarity symmetries as gradual invariances are also addressed by Schwarzfischer [68].

5. MATHEMATICS DIDACTICS: LEARNING TO LEARN

As is well known, the word "mathematics" is derived from the ancient Greek term *mathēmatikē téchnē*, which can be translated as *the art of learning*. However, learning itself is by no means limited to the deductive derivations that the epistemological *rationalists* admired for their logical necessity — on the contrary.

5.1. All Learning is Stochastically Based

The example in Figure 1 shows that enactive and iconic aspects occur simultaneously in the learning process. These correlate to varying degrees. If the similarities of occurrence are sufficiently large, the two sub-systems are coupled, cf. Arvid [69] and Wolfgang [70]. This means that the two aspects (e.g. the iconic-cognitive and the enactive-motor) are combined to form a unit of *action knowledge*.

According to Arvid [69] "the acquisition of action knowledge [...] takes place *automatically*, i.e. regardless of whether there is an intention to learn or not, and regardless of whether attention is directed to the movement-effect relationship or not." From an evolutionary and developmental psychology perspective, this is not possible otherwise anyway. This is because the prerequisites for higher-level attention and a conscious intention to learn are only the result of this development and therefore cannot already be presupposed.

An example of the unconscious linking of internal and external perception is provided byHenri [71] andHenri [72] who investigates the problem of spatial vision. Poincaré ascertains that the experienced three-dimensionality of space cannot be explained by binocular vision alone. He comes to the conclusion that the unconscious contraction of the eye muscles must be included in the calculation in order to explain spatial vision. Iconic perception (as exteroception) is automatically and unconsciously linked here with the enactive perception of one's own body activity (as interoception). Such unconscious processes are referred to as *ratiomorphic* cognitions, cf. Bischof [33].

5.2. Cognitive Modelling Through Active Inference

In fact, the supposedly *datum* (lat. the given) is a result of unconscious, ratiomorphic processes and consequently a *fact* (lat. a made, something made by an action). ***Positivism***, which dominated the discussion of scientific theory for a long time, also took simple perceptions of form as given. Positivism ignored the many unconscious conclusions on a microcognitive scale because they did not correspond to its methodological dogmas.

Many unconscious perception processes lead to a cognitive model that represents a small section of reality, see Schwarzfischer [46]. The key point is that we need to interpret this model in terms of action theory. From an evolutionary and developmental psychology perspective, we are not passive creatures that only become active in response to external stimuli, as assumed by classical cognitivism with its *input-processing-output* scheme. This is also known as the *sensorimotor* schema, because the action (motor activity) is interpreted as a secondary reaction to a primary perceptual input (sensory activity). According to Prinz [70] the empirical sequence should be conceived in the opposite way: His *ideomotor* approach assumes that humans are active beings who already perform exploratory actions prenatally. These explorations continue throughout childhood and into adulthood.

The action is therefore primary and only produces certain perceptions as a secondary effect. For a completely passive-contemplative observer, who does not even move his eyeballs, the majority of his world remains hidden. This consideration led Kirsh and Maglio [73] to distinguish between *instrumental actions* and *epistemic actions.*[6] In each case, the focus is on a different aspect, even if the plot might look very similar from the outside. An example illustrates this: Opening the fridge to pour milk into my coffee, is an instrumental action. On the other hand, it is an epistemic action if I open the fridge before shopping to see if there is any milk left. In the first case, I want to change a state of *affairs in the world* (the taste of my coffee). In the second case, I only want to change my *knowledge about the world.* Accordingly, epistemic actions are a central method of developing the self and the world model. In fact, even the kicking of an infant makes self-development possible, because this is the only way to create a body schema. As already recognized byHenri [71] a passive analysis of perceptual data is not sufficient.

A similar combination of *enactive cognition* and *iconic cognition* can be applied to the example of Figure 1. It then becomes clear that we perform an unconscious, *mental rehearsal* of displacement when looking at it, which is an epistemic action. The properties of an object are tested through virtual trial and error. This reveals similarities with early childhood learning experiences based on real objects, which are real because the child can act on them — see Schwarzfischer [46].

What we believe to see immediately in Figure 1 is therefore the *process result* of extensive learning experiences. Without these learning experiences, not even the basic perceptions that we take for granted can function, as Held and Hein [74] were able to demonstrate in their famous study.[7] In "Going Beyond the Information Given", Bruner [75] emphasizes the necessity of overcoming the existing information of the accidental factual — and he even claims the impossibility for cognitive systems not to go beyond the existing information. As Figure 1 shows, the

[6] In their essay, Kirsh and Maglio [73] distinguish between "epistemic actions" (which aim at a better knowledge of the world) and "pragmatic actions" (whose aim is to achieve a certain state of the world). From a semiotic point of view, it is unfortunate when they speak of "pragmatic actions", because of course "epistemic actions" also have their pragmatics. That is why we speak here of "instrumental actions" instead of "pragmatic actions".

[7] In the famous study by Held and Hein [74] young cats were reared in complete darkness. They could only see under controlled conditions, whereby two groups were formed: One cat moved actively and the other cat was only moved passively by a device. The visual impressions in these bright learning phases were identical for both animals, as they moved through the same scene in the same way. But the cats that were only moved passively did not learn to see. They moved in the test phase as if they were blind. In relation to our example from Figure 1, this means that the microcognitive processes must also be learned under certain conditions. One can certainly speak of active inference (i.e. "active inference" on the basis of action effects), although there is still no reflected consciousness in the narrower sense.

perception of every Gestalt already goes beyond the positively given. This is because the expectation that the circle simply continues “under” the square is not contained in the sensory data. This expectation therefore represents something qualitatively new, which characterizes the understanding of the observer. Only the conscious or unconscious construction of a cognitive model that goes beyond the given can be interpreted as understanding. Schwarzfischer [46] following Jean Piaget, describes it as *decentering* when the scope of validity of a cognitive model is extended.

In recent cognitive science, *active inference* plays a decisive role, see Hohwy [6]; Arvid [69]; Wolfgang [70]; Andy [76]; Friston, et al. [77]; Burr and Jones [78] and Parr, et al. [5]. Expectations for the effects of future actions are derived from previous experiences. What specific perceptions can be expected from certain motor activities? This linking of motor and sensory patterns enables the infant to distinguish their own body from the environment (e.g. when I specifically raise my arm and see it in front of my eyes). Distinguishing whether the world is moving or whether I am moving is also only possible through feedback with the motor signals (through the so-called *reafference*).

The basis for this is a cognitive model from which the expected effects are deductively derived. The model itself is created and constantly modified by the underlying mechanics of Bayesian statistics, see Parr, et al. [5]. This is because every movement of the observer (e.g. an eye movement), no matter how small, changes the sensory input and at the same time the knowledge about the world. The distinction before the eye movement and after the eye movement must therefore be used productively. This is possible in the Bayesian approach if a distinction is made between *prior* and *posterior*. This turns every eye movement etc. into an *epistemic action* that checks and corrects the expectation (as *prior*) (and thus makes it *posterior*). The general approach according to Karl Friston is based on minimizing *prediction errors*. A hierarchical *multi-level model* is used, which works in parallel at several spatial and temporal granularities and also relates these levels to each other.

5.3. The Cognitive Model is Falsifiable

If human vision worked like a camera image, then different viewers would always perceive the same image. But perception is more than just the technical scanning of pixels provided by a sensor, as Alan [79] shows. This is because the brain is responsible for the majority of perceptual processes and the eye for only a small part. Accordingly, previous experiences change these brain processes and consequently also the result of the perception process. Two observers can therefore perceive quite differently without necessarily being aware of it themselves. This is because people only know the results of their *own* perceptual processes — the results of *other* observers are not available to them for comparison. How can a person nevertheless check a cognitive model that they have formed on the basis of a perceptual scene?

The traditional view of philosophy understands observable facts as statements, cf. Alan [79]. This is quite plausible, because even every measurement result has the logical structure of an if-then proposition, as Peter [19] deconstructs. However, not only a formal measurement has this structure. If we follow Karl Friston’s *predictive mind* approach, then this applies to every action, see Hohwy [6]. This applies to instrumental actions anyway, because they already presuppose a goal. But even unconscious perceptual actions (e.g. spontaneously turning my head because I heard something from the right) are guided by expectations (e.g. that something will now be seen that caused that sound). All epistemic actions are motivated by an expectation that thematizes possible outcomes based on an interest in knowledge.

According to Friston, et al. [77] these implicit or explicit expectations are constantly updated. For example, they are already different after an eye movement than they were before it. Therefore, even eye movements are to be understood as epistemic actions that enable Bayesian inference, cf. Friston, et al. [80]. The same naturally also applies to head movements and the like. The expectation can easily change — or turn out to be completely wrong. The following example illustrates this.

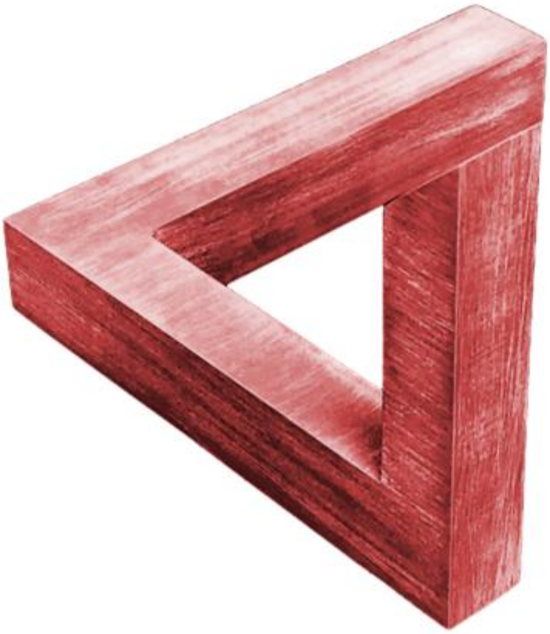

Figure 2. The so-called "Penrose triangle".
Source: Struckduck [81].

In Figure 2, we see the so-called *Penrose triangle* as a spatial model. This is not trivial because the Penrose triangle is an object that can be represented in a perspective drawing but cannot exist as a solid object. The Penrose triangle (sometimes also called a *tribar*) is a so-called "impossible figure". It shows three beams that appear to be at right angles to each other and yet are connected to form a triangle. It thus violates several laws of Euclidean geometry — including the fact that the sum of the interior angles in a plane triangle is always 180°. The viewer is confronted with the difficulty of having to constantly reinterpret their distance from the parts of the tribar and their position in the space depicted.

To understand *falsification*, this Penrose triangle is a good example. This is because when we perform epistemic actions, we experience something that completely contradicts our expectations. Figure 3 shows a sequence of how our perception would change if we were to rotate or move around this Penrose triangle.[8]

Figure 3. The Penrose triangle as a model for a 3D-printer.
Source: Struckduck [81].

[8] A video documentation of the rotation can be found at Struckduck [82].

The sequence in Figure 3 clearly shows that this Penrose triangle is a fake triangle. Only from one single perspective it appears to be a real Penrose triangle as a 3D object. From all other perspectives, the picture is completely different. The expectation that the object would also be a Penrose triangle from other angles is revealed to be a mistake. This expectation could be falsified with an epistemic action. Accordingly, the cognitive model that was formed on the basis of Figure 2 proved to be incorrect or at least incomplete.

The example of this Penrose triangle makes it abundantly clear what happens constantly in everyday life: A cognitive model is formed on the basis of previous experiences and sensory input. An expectation is deductively derived from this and tested through epistemic actions. Every epistemic action can be understood as an experiment, irrespective of whether it is carried out in a consciously reflective or in an unconsciously automated manner. According to Karl Friston's *predictive mind* approach, the central issue is whether expectations are formed or not, see Hohwy [6].

5.4. A Consequent Processualization of Knowledge

As the examples in Figures 1 to 3 demonstrate, perception and cognition must each be conceptualized as *process outcomes.* The traditional view is deceptive, as these are not propositional phenomena whose truth content can be adequately formulated in static sentences. On the contrary, it can be assumed that dynamic probabilities rather than truths are appropriate for description. This has already become clear in the analysis of Figure 1, because evolutionary and individual development must also be taken into account there.

But the development does not end with the observation or reconstruction of a scene. Only when we recognize current knowledge as an *interim report* of a dynamic process do we come close to the essence of cognition as active inference. This is because our next eye movement already brings new insights by bringing into focus something that was previously only blurred in the periphery. Accordingly, details become visible there that were previously only modelled ratiomorphically as probabilities: A *potentiality* becomes an *actuality.* But only the subsequent embedding in a pragmatic context assigns it a semantic role. Consequently, any semantic role is initially only present as a potential before it is realized.

The learning history of an individual is ultimately similar to the training of an artificial intelligence of today's design. For both begin as a *tabula rasa*, so to speak, if we disregard the structural learning ability that we must always take for granted. "*The baby, assailed by eyes, ears, nose, skin, and entrails at once, feels it all as one great blooming, buzzing confusion*", as William [83] puts it. Strictly speaking, there are no propositions that can be clearly assigned the attributes "true" or "false" — neither for the newborn child nor for us adults.

In everyday life, we are not able and do not always want to carry this consistent processualization with us, as it would exceed our cognitive capacities — see Gabriel [84]. That is why we act *as if* things were simpler. In this way we avoid an infinite regress that would make us completely incapable of any action (which in turn would have prevented our evolutionary development). In his *Philosophy of the As-If*, Hans [16] describes how widespread and useful this illusion is in the most diverse areas of life and science. The science theorist van Fraassen [12] also abandons the illusion of absolute truth when he postulates: "Science aims to give us theories which are empirically adequate; and acceptance of a theory involves as belief only that it is empirically adequate." We must therefore be content with a sufficiently good approximation of empirical phenomena. And this only means that we develop cognitive models and theories that allow us to formulate specific expectations in the form of predictions that have an acceptable measurement uncertainty. For cognitive systems in general, Karl Friston's approach goes in a similar direction. He formulates the *prediction error minimization* as a central cognitive mechanism in order to minimize the discrepancy between model and world — see [5].

6. MATHEMATICS DIDACTICS: LEARNING TO LEARN HOW TO TEACH

6.1. Mathematics Didactics from a Processual Perspective

Consistent operationalization and thus processualization could actually be a matter of course for mathematics didactics — but it is not. Although the term "didactics" derives from the ancient Greek *didáskein* for "to teach", the verb often disappears behind a rather static view that can be described with the noun "information design". This is because in didactics, didactic artifacts are often confused with the processes of design itself. For example, textbooks or PowerPoint presentations are then considered to be the object of didactic design — and not the cognitive processes of the learners, to which the attention of didacticians should actually be directed. If we understand didactics as a transdisciplinary design discipline, then we can carry out a consistent processualization. Schwarzfischer [85] points out the differences between the operationalizations of the type *intra-object*, *inter-object* and *trans-object* in the late work of Jean Piaget in order to open up the process perspective.

In the first phase *intra-object*, the observer only knows isolated facts or objects and analyzes them without any connection between them. (Example: Music of different styles and eras is known. However, the different pieces are seen or heard without any relationship to each other). In the second phase *inter-object*, individual transformations are known, how one fact can be transformed into another. (Example: The known pieces of music can be transformed into each other by changing the tempo, rhythm and, above all, the timbre or instrumentation etc.). Only in the third phase, *trans-object*, are those alternatives available about which there is no empirical knowledge — all possible cases can be logically deduced, even if they have never actually occurred. (Example: Through an understanding of Gestalt perception, which underlies musical perception, every compositional parameter can be specifically modulated.)

As is easy to see, *intra-object* typically involves processes of categorization. Here, gradual invariances are used as similarities between the elements to form a category. In the simplest case, this concerns different perspective views of the same object.

This leads to a cognitive model of the object that spatially integrates these views in such a way that previously unknown views are assessed as to whether they fit the cognitive 3D model or not. In addition, mental rehearsals using the model become possible (e.g. mental rotations to deductively derive new views). These processes can occur in all gradations from ratiomorphic-unconscious to rational-conscious. Irrespective of this, the character of an action is at least implicitly assumed, because the processes certainly pursue identifiable goals. Since every perception can already be understood as a selection action, Schwarzfischer [46] interprets any observation as an action of an *embodied observer system*.

Consequently, the transformations of the second stage *inter-object* are also to be conceptualized as actions, regardless of whether they are carried out in reality or as mental rehearsal actions. The same applies to the third phase *trans-object* and the opening up of the *space of possibilities* of corresponding transformations.

This allows us to interpret all processes of observing, learning and teaching as actions. Mathematics didactics can therefore be built on a cognitive-constructivist foundation, because both mathematics and didactics are based on transformations. Not only mathematical operations in the narrower sense are meant here. We have already pointed out the relevance of epistemic actions above. It is of central importance for learners not to sit passively in front of a mathematical task — especially if it is not immediately obvious. Rather, the task must be actively acquired, even if for some learners this first becomes an excursion into the art of guessing or guesswork. If the scheme of the task is not immediately recognized, the solution approach cannot simply be deduced. Through explorative behavior and abductive reasoning, a solution approach can still be found by looking at the parts of the task in different granularity.

Generally speaking, intelligence is more of an action pattern than a static character trait. This applies equally to artificial intelligence as it does to human intelligence. In cases of uncertainty (i.e. when the cognitive subject does not know what to do), we should know what to do (to get out of the uncertainty).

Thrun and Norvig [86] put this in a nutshell: "Artificial Intelligence is the technique of uncertainty management in computer software. AI is the discipline that you apply when you want to know what to do when you don't know what to do."

However, too little uncertainty can also have a paralyzing effect: In order to learn something new, we have to say goodbye to apparent certainties. Every time we learn something new, we have to unlearn something old. For example, we have to let go of the idea that the world can be described using objective (and therefore static) probability distributions.

Only in this way can we gain an understanding of the subjective-dynamic probability assessments that are not primarily used to capture the world, but rather the gradual-continuous accommodations of the cognitive models: It is not the world that is changed, but the cognitive model of a section of reality.

Didactically, the adaptation of a cognitive model can be stimulated as an adaptive process, but there is no causal effect from the teacher, see Kersten [87]. The relationship between teacher and learner is therefore itself to be understood as stochastic.

The teacher can try to deliberately irritate the learners — so that the latter reconstruct their cognitive models in order to integrate the irritation that has occurred. This follows the basic principle of irritating the course of action, which has already been introduced in section 4.2. In a successful case, this triggers an *Inquiry Cycle* according to Dewey [49] which in turn integrates abductions, deductions and inductions, see Kersten [87] and Peter [88].

6.2. On the Pragmatist Philosophy of Mathematics

The learner's perspective is fundamentally an individual lifeworld. It is therefore characterized by subjective experiences, which can only be the subject of collective coordination processes in the next step. It is therefore the results of individual cognitive processes that can serve as the content of communication. We arrive at this point of view when we distinguish, with Siegfried [17] and Siegfried [18] between *action (process)*, *actor/actant* and *action result*. It has already been made clear above that we can interpret every observation as a perceptual action. Siegfried [17] and Siegfried [18] then describes these three observation perspectives more generally as *process (progression)*, *process carrier* and *process result*.

If we follow this logic, we can thematize every Gestalt-like "something" as a process result. The corresponding course of the process is to be understood as *actual genesis*, whereby the individual's cognitive processes act as process carriers — see [46].

The cognitive processes only become apparent if we choose a finer granularity of analysis than would be sufficient for observing the results. This is why we can also speak of *microcognitive processes*, which often take place unconsciously and ratiomorphically. This is because a coarser granularity is sufficient for our everyday consciousness — and a finer resolution would tie up more cognitive resources, which in turn would not have been useful in evolutionary terms: For our ancestors, it was sufficient to flee when a predator appeared (as a process result of microcognitive perception processes). Anyone who spent too long with the processes of unconscious cognition at this moment simply did not survive, see Schwarzfischer [46]. Consequently, it was a survival advantage to be content with a rather coarse granularity, which also ties up fewer biological resources because the neuronal effort is lower.

As a science, mathematics is not primarily used to cope with our everyday lives. The philosophy of mathematics reflects, among other things, its epistemological basis. As Wilhelm [89] explains, the philosophy of mathematics has historically moved from a Platonic position (according to which the objects of mathematics have an independent existence in the world as "eternal ideas") to increasingly constructivist views (according to which the objects of mathematics are the results of social and cognitive construction processes and cannot simply be ontologically presupposed or axiomatically postulated).Wilhelm [89] summarizes three paradigmatic positions in an overview table:

1. As a *rationalist*, Gottfried Wilhelm Leibniz presupposes the truths of reason as *necessarily* true. He assumes that these truths, which always already exist, are discovered through the analytical method.

2. The *transcendental philosopher* Immanuel Kant represents a position that seeks to mediate between rationalism and empiricism. He shows that both must be understood as necessary, complementary parts of the cognitive process. The propositions of mathematics are therefore not simply *analytical* and *a priori* true, as Leibniz claimed. According to Kant, mathematical propositions are *synthetic* and *a priori* true. This means that not only *analytical* truths are discovered that are already implicitly contained in the grammatical subject of a propositional statement (e.g. that a sphere must necessarily be round). In the case of *synthetic* statements, the predicate expresses something that is not already contained in the subject of the proposition (e.g. if the sphere is also transparent). Kant is of the opinion that mathematics only contains propositions that are *synthetic a priori*. He therefore assumes that these truths already exist before their discovery and are not dependent on empirical experience (i.e. are *a priori* in relation to experience).

3. As a consistent *empiricist*, John Stuart Mill goes one step further and claims that the theorems of mathematics are *synthetic* and *a posteriori* true. According to Mill [11] the axioms from which other mathematical truths are deductively derived also require justification. However, only sensory perception and inductive logic would be available for this. Consequently, according to Mill [11] mathematics is not a deductive, but ultimately an inductive science — see also Büttemeyer [90].

From a didactic point of view, this is not a triviality, as Mill [11] continues this assumption into the consequences of developmental psychology. For him, the concept of number is not an eternal Platonic entity, but the result of inductive processes in an individual's cognition. According to Mill [11] numbers and thus also arithmetic are ultimately based on the specific experiences of a concrete individual, who combines and abstracts these experiences into concepts of number and quantity — see Neunhäuserer [91]. The approach of Mill [11] is therefore very similar to the more recent concepts of *Radical Constructivism*, such as those developed by Von Glasersfeld [92].[9] Here, too, the active role of the observer is emphasized, who creates the entities of his observation through constructive cognitive acts.

In line with this, Schwarzfischer [46] formulates: "Speaking of parts already presupposes the punctuation of continua, whereby punctuation is to be thought of *against the background of other possibilities*. For this reason, punctuation is defined in Schwarzfischer [68] as a *basic pragmatic operation*."

Accordingly, a syntactic analysis (on which any stochastic evaluation is based) can only be carried out after it has been determined what is to be considered a part or an element in the first place. At least the arbitrary choice of granularity must therefore always be presupposed (even if the pragmatic act of this choice often does not penetrate consciousness — and thus turns out to be a ratiomorphic process).

[9] See also the visualized examples of the distinction between *conception* and *perception* with regard to the concept of *number in perception* in Ernst von Glasersfeld [93]

Consequently, no analysis is purely objective because it cannot work without the arbitrary act of an observer-subject. Implicit in the *choice* of granularity and the segment of reality to be observed (which we had defined as a *pragmatic basic operation*) is already the result of the analysis — even if we do not yet explicitly know (or even can know) this result. The findings from this analysis are therefore *synthetic* (because they extend our knowledge) and *a posteriori* true (because they required the empirical intervention of at least one basic pragmatic operation). Which and how many entities can be observed in reality or in a mental rehearsal depends on these premises of observation.

Consequently, mathematics is a cognitive construction of the observer, as conceived by Von Glasersfeld [92] with recourse to Jean Piaget. Mathematics didactics is therefore also based on acts of observation and communication.

Other modes of observation would lead to different observations and thus to different results. Strictly speaking, their basic pragmatic operations would always have to be named, because the derived assertions are only true under these (abductively obtained and inductively confirmed) premises. Such mathematics would be possible, but no longer really practicable.

As Neunhäuserer [91] emphasizes in his critique of *intuitionism* (as a well-known variant of cognitive-constructivist mathematics), this approach would make mathematical practice much more complex without bringing any recognizable advantages in terms of mathematical possibilities. Wilhelm [89] argues similarly and shows why axioms are no longer completely dispensed with in the *Erlangen School of Constructivism*. Insisting on ever more finely resolved constructions and reconstructions would be tantamount to an infinite regress and would paralyze mathematical practice.

6.3. Conclusion: Operational Stochastics is Unavoidable

These statements show that uncertainties cannot be avoided — unless we exclude them *dogmatically*. And axioms, which must not be questioned any further, are nothing more than a dogmatic basis for deductive inferences. The term “dogma” is to be understood here in its ancient Greek meaning (*dógma*) for “doctrine” or “decision”. In order to preserve our ability to act in everyday life or in mathematics, we are better off ignoring the infinite chain of ratiomorphic-unconscious processes of microcognition.

However, when we look at the details, our everyday life is not deterministic, but complex-creative-stochastic. This begins with the simplest perceptions: Without an unconsciously operating cluster analysis, our visual perceptual field would be just the *great blooming, buzzing confusion* as that William [83] described it.

And even if our episodic memory is not sufficient to remind us of the specific learning experiences in our earliest childhood, we can assume that we have constructed a number concept empirically through individual experiences over time.

The use of invariants in the perceptual field forms the basis for this. Here we must assume active use through epistemic actions, as described by the ideomotor approach of Parr, et al. [5]. The mathematical objects are therefore abstractions of physical objects. And these in turn were inferred from the changes in sensory input, which were actively induced by epistemic actions — cf. also Wolfgang [70]. These cognitions presuppose an embodied and situated observer who is not limited to passive-contemplative observations. Accordingly, any cognition can consequently be described as an *enactive cognition*, as illustrated by the example in Figure 1.

However, because as humans we are situated in a complex environment, our sensory input is always affected by a variety of influences. The world is therefore not a place of unambiguity, but a space of possibilities where causalities only reveal themselves as probabilities. These probabilities can be modeled, for example, as causal maps, which can be implemented mathematically as Bayesian networks. Russel [1] uses a programming language called

BLOG (an abbreviation for Bayesian Logic) in his research group for artificial intelligence. There is also growing evidence in cognitive developmental psychology that Bayesian statistics can be used to reconstruct the learning of young children — see Gopnik [94] and Tabor and Burr [95].

Consequently, operational stochastics is completely unavoidable for understanding people. Accordingly, this mathematical field could have a far greater appeal if didactics would integrate the concrete world of life more strongly. It should become clear that each everyday action can be understood as the *art of conjecture*. Examples from perceptual processes would be particularly suitable for this, because they can maximize vividness. The emphasis could actually be placed on processuality in order to bring the dynamics of developments more into focus. Such change processes can have very different granularity on the time axis: The extremely slow processes of cognitive development are typically specified in changes that occur from month to month. In contrast, the dynamics of perceptual processes take place in seconds (e.g. when Sherlock Holmes enters a room and registers its mood by taking in new information every second, which immediately turns the previous *a priori* assessment into an *a posteriori* opinion, etc.). Motor processes (e.g. finding and maintaining balance on an irregularly swaying floor) take place even more quickly.

It would therefore be possible to explore this exciting field with childlike curiosity — without the fear or boredom mentioned at the beginning of this paper. However, this would require us to break away from traditional schemes and examples that are far removed from real life: No restriction to "methodological artifacts" of stochastics lessons, just because these can be conveniently translated into school grades in the true/false scheme. And supplementing rather static examples (such as the famous *false-positive* and *false-negative* diagnoses in medicine and law) with dynamic applications in cognition, artificial intelligence and robotics. This would perhaps finally bring the attention of learners to stochastics— and to contemporary cognitive science.

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